CHAPTER 03

Vectors and Graphing

3.1 VECTORS AND SCALARS

To completely specify some physical quantity it is not sufficient to just state its magnitude. In the previous chapter you saw that to describe the motion of an object you needed to state its speed and the direction it was heading. In other words, its velocity. Displacement and acceleration needed a direction too. They were all called **vector** quantities. The word 'vector' comes from the Latin *vectus* meaning 'to carry' — a word implying direction. In biology a vector is an organism that carries disease from one place to another; for example, a mosquito is the vector for malaria. In physics it means a quantity that needs both magnitude and direction to specify it fully. This chapter continues the discussion about the nature of vectors as used in physics. Later in the chapter there is a discussion on graphing and how graphs can be used to solve problems.

Some of the questions that puzzle students about vectors and graphs are:

- Can two vectors having different magnitudes be combined to give a zero result? Can three?
- Can a vector have zero magnitude if one of its components is not zero?
- If time has magnitude and a direction (past → present → future), is it a vector?
- How can a statistician look at an unemployment graph and say the unemployment rate is increasing whereas another statistician can say the rate is decreasing?

Table 3.1 Some scalar and vector quantities

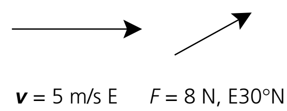
SCALAR	VECTOR
Length	displacement
Speed	velocity
Time	acceleration
Volume	force
Mass	weight
Energy	momentum
Frequency	torque
Pressure	moment
Power	electric current
Temperature	electric field
Charge	magnetic flux density

Photo 3.1
XYZ Plotter. A computer controls the X, Y and Z coordinates of the cutting head in a tool-maker's workshop. A steel mould is cut and used to make plastic parts by injection moulding.

Representation of vectors

A vector quantity can be represented by an arrow called a vector. The length of the arrow represents the magnitude of the vector quantity, and the direction of the arrow shows the direction of the vector quantity. For example, Figure 3.1 shows two vectors representing the vector quantities velocity and force:

Figure 3.1



NOVEL CHALLENGE

A ranger at Mt Mungo National Park published an booklet entitled Twenty Family Walks. In the introduction he wrote, 'The walks are short, ranging from a kilometre and a half to five kilometres; the average is two and a half kilometres.'

A What is the total length of all twenty walks?

B What is the greatest possible number of walks more than 4 km long?

C If there are three walks of 5 km each, what is the greatest possible number of walks shorter than 5 km but longer than $2\frac{1}{2}$ km?

Note: **wind directions** are confusing. A wind direction is where the wind is coming *from*. For instance, a south-easterly breeze comes *from* the south-east (S45°E or E45°S) but is *heading* north-east. Be careful to draw your diagrams carefully when wind directions are mentioned.

Scalar quantities require no statement about direction. For example, time = 3.5 s, mass = 25.5 g and current = 2.0 A are scalar quantity measurements — no direction has to be specified. The word 'scalar' comes from the Latin *scalaris* meaning 'pertaining to a ladder'. This refers to the stepwise change in the size of something without any reference to direction. *Note:* in maths class you may be taught how to work with 'unit vectors' using the symbols **i**, **j** and **k**. You could still use this system in physics if you like but it won't be discussed further in this book.

Activity 3.1 Finding your way home

Photocopy a map of your local area (e.g. a street directory). Draw in the route you normally take to school and estimate the distance travelled. Draw a straight line from your home to the school and determine your displacement or distance 'as the crow flies'. Include the direction.

VECTOR MANIPULATION 3.2

There are many cases in the world around us where more than one vector quantity is involved. When this is the case, we need rules to perform some form of arithmetic. We apply normal rules to scalar quantities — rules for addition, subtraction, multiplication and division. In the world of vector arithmetic, these rules must also take into account direction of the vector quantities. If you go 3 m N and then 4 m E your displacement is not 7 m. You'll see why below.

In this book we will represent a vector by printing it in ***bold italics***. For example, vector *A* will be represented as ***A*** and vector *v* as ***v***. Some books and teachers may prefer to underline the vector with a tilde (~), e.g. $\underset{\sim}{A}$, $\underset{\sim}{a}$, $\underset{\sim}{v}$ instead of using bold.

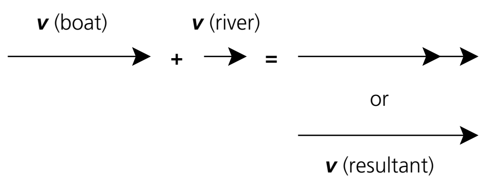
Vector addition

Two or more vector quantities can be combined to produce a single **resultant** vector.

Case 1

Consider rowing a boat at 5 m s^{-1} E in water that is also moving E at 1 m s^{-1}. Your actual velocity is 6 m s^{-1} E and is found by placing the two vector arrows head-to-tail. The resultant is a line drawn from the tail of the first arrow to the head of the second arrow (Figure 3.2).

Figure 3.2



Note: when *adding* vectors they should be placed *head-to-tail* and the resultant will always start at the tail of the first arrow and end at the head of the second arrow.

Case 2

Consider the same boat being rowed *against* the current. In this case the velocity of the river is 1 m s^{-1} W and is in the direction opposite to that of the boat and hence will slow the boat down:

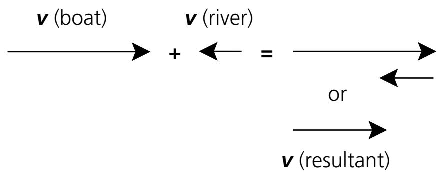


Figure 3.3

The resultant velocity is 4 m s^{-1} E. Note that when two vectors in the same line are added, the resultant has a direction the same as the larger vector.

Case 3: Vectors not in a line

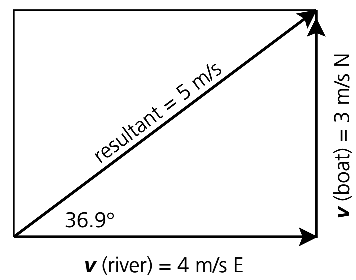
Imagine you are rowing north at 3 m s^{-1} across a river but the river current is flowing east at 4 m s^{-1}. You would be dragged off-course by the current and your resultant velocity would be 5 m s^{-1} (Figure 3.4). Note again that the two vectors are added head-to-tail. The resultant is a line drawn from the tail of the first arrow to the head of the second arrow. This resultant can also be drawn as the diagonal of the parallelogram constructed by using the two given vectors as sides. Either method is acceptable.

The solution to this problem is in two parts — a magnitude component (5 m s^{-1}) and a direction component (E36.8°N). This is achieved in the following manner:

1 *Magnitude* Because the starting vectors for the boat and river are at right angles (N and E), Pythagoras's theorem can be used. Resultant = $\sqrt{4^2 + 3^2}$. If a scale diagram was used, the resultant could be measured with a ruler.
2 *Direction* Because the two vectors and the resultant form a right-angled triangle, trigonometry can be used: i.e.

$$\tan \theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{3}{4} = 0.75, \text{ hence } \theta = 36.9°$$

Figure 3.4



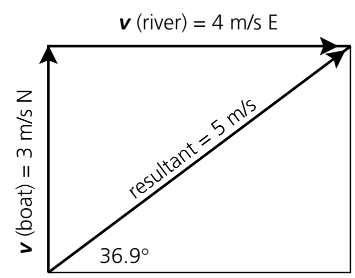
Note that the order of addition is not important. Figure 3.4 could also be drawn as shown in Figure 3.5. The resultant is still the same.

Refresher The trigonometric ratios for the right-angled triangle shown in Figure 3.6 are given below.

$$\sin \theta = \frac{\text{opposite side length}}{\text{hypotenuse}}\,;\ \cos \theta = \frac{\text{adjacent side length}}{\text{hypotenuse}}\,;\ \tan \theta = \frac{\text{opposite side length}}{\text{adjacent}}$$

Hint: when the calculator displays 0.75, usually you need to press 'shift' then either sin, cos or tan to convert this to the angle. *Note:* Latin *sinus* = 'curve'.

Figure 3.5



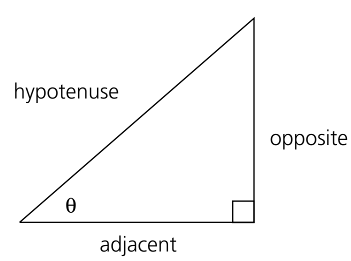
Questions

1 Calculate the values of θ in the following right-angled triangles (do not write in this book):

Table 3.2

	ADJACENT	OPPOSITE	HYPOTENUSE	RATIO	θ
(a)	10	7		tan θ =	
(b)	8		13	cos θ =	
(c)		9	20	sin θ =	
(d)	200	50		tan θ =	

Figure 3.6



Note that the ratios for sin and cos are always 1.0 or less; only tan can go beyond ±1.0. If you disagree with the values of θ shown in the back of this book, check that your calculator is in degrees (shown by a small DEG in the display). A common mistake occurs when the calculator is put into radians (RAD). Ask someone near you or your teacher if you get stuck.

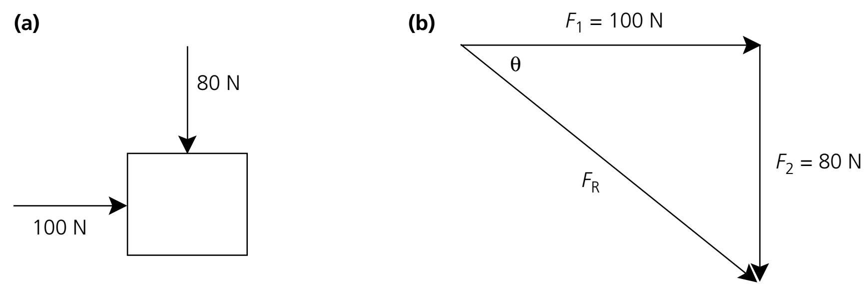
Example 1

Two forces act on a crate as shown in Figure 3.7**(a)**. Calculate the resultant force.

Solution (See Figure 3.7**(b)**)

Resultant ($\boldsymbol{F}_R$) = $\sqrt{100^2 + 80^2}$ = 128 N. The angle θ is found by $\tan\theta = \frac{80}{100}$, so θ = 38.7°.

Figure 3.7



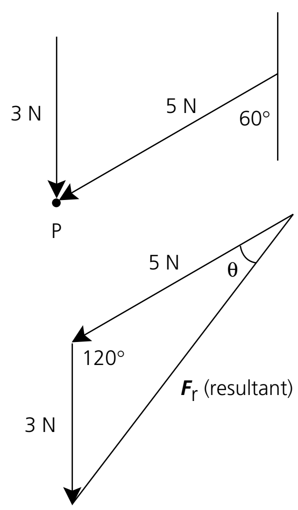
Case 4: Vectors not at right angles

In such cases, Pythagoras's theorem and the three trigonometric formulas do not apply. However, there are several other solutions you may use. Check with your teacher whether you are expected to use these additional formulas:

Method 1 If you have the value of one angle and its opposite side use the sine rule:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Figure 3.8



Method 2 If the sine rule won't work and you have two sides and one angle you may be able to use the cosine rule if you are familiar with it:

$$c^2 = a^2 + b^2 - (2ab \cos C)$$

Method 3 By dropping a perpendicular from one apex to the opposite side. This produces two right-angled triangles, which may possibly be solved using Pythagoras's theorem or trigonometry.

Example 2

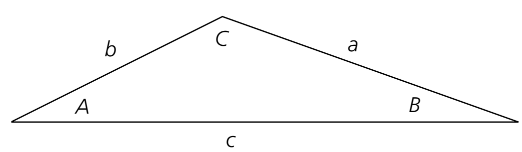
A force of 3.0 N south and 5.0 N S60°W act on the same point (Figure 3.8). Determine the total force that must be acting on that point.

Solution (See Figure 3.9)

The cosine rule must be used to determine the magnitude of the resultant force:

$$\begin{aligned} \boldsymbol{F}_r^{\,2} &= 3^2 + 5^2 - (2 \times 3 \times 5 \times \cos 120°) \\ \boldsymbol{F}_r &= \sqrt{49} \\ \boldsymbol{F}_r &= 7\text{ N} \end{aligned}$$

Figure 3.9



Use the sine rule to determine the direction:

$$\frac{\sin\theta}{3} = \frac{\sin 120°}{7}$$

$$\therefore \sin\theta = \frac{3}{7} \times \sin 120°$$

$$\theta = 22°$$

Total force = 7 N, S38.2°W.

NOVEL CHALLENGE

Is it best to run or walk in the rain if you want to stay as dry as possible? Imagine you had to get from one building to another and the rain was falling vertically (for simplicity). *What are the benefits and disadvantages of running versus walking? So what should you do?*

Questions

2 Find the magnitude and direction of the resultant vector obtained by adding **(a)** displacements of 30 m E and 20 m N; **(b)** velocities of 16 m s^{-1} W and 30 m s^{-1} S; **(c)** forces of 20 N NW and 10 N NE.

3 Calculate the resultant force when the forces listed below act together on a wooden log: **(a)** 18 N horizontal and 24 N vertical; **(b)** 2.5 × 10–3 N east and 1.8 × 10–3 N north; **(c)** 300 N horizontal (pulling) and 150 N at an angle of 25° to the horizontal (pulling).

4 A red cricket ball of mass 150 g is thrown upwards with a velocity of 20 m s^{-1}. Gravitational acceleration allows the ball to reach a maximum height of 20 m in a time of 2.0 s, after which it falls to the ground and strikes it 2.0 s later. From this statement, name three scalar quantities and three vector quantities.

5 A force of 18.0 N south and 14.0 N S50°W acts on the same point P. Determine the total force acting on that point.

Vector subtraction

If your mass was 65 kg and after the Christmas holidays you had gained 5 kg, your mass would of course be 70 kg. You could say that your **change in mass** was +5 kg. If, instead, you dieted over Christmas and lost 5 kg your final mass would be 60 kg. Your change in mass would be –5 kg. You could have worked this out in Grade 8. In physics we need to be very particular in the way we talk about 'change' in a measurement, particularly vectors. In physics 'change' means difference, but with this convention:

Change in a measurement (Δ) = final measurement – initial measurement

After dieting: change in mass (Δm) = final mass – initial mass
= 60 kg – 65 kg
= –5 kg

This is simple for scalar quantities like mass, temperature and bank balances. But in physics it is also necessary to subtract vectors. In maths, you should have learnt that subtraction is the same as adding a negative, that is, 10 – 8 is equivalent to 10 + –8 and the answer is +2 either way. When subtracting vector ***B*** from vector ***A***, the direction of vector ***B*** is changed to its opposite and then added to vector ***A*** (head to tail).

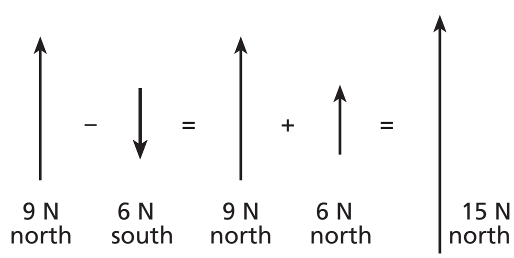


Figure 3.10
Subtraction of vectors.

Example 1

A ball falling vertically strikes the ground at a velocity of 10 m s^{-1} (downward) and rebounds vertically upward at a velocity of 8 m s^{-1}. Calculate the change in velocity. Assume the downward direction is negative.

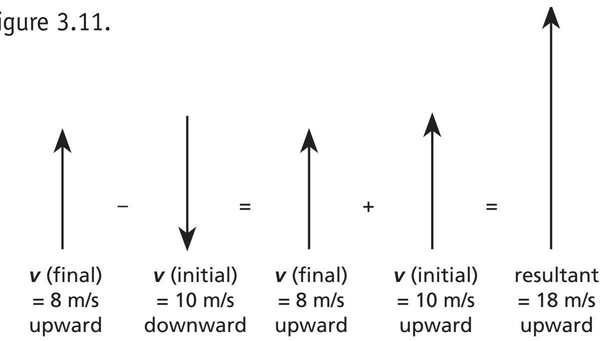
Solution

'Change' means difference, hence:

$$\begin{aligned}\text{Change in velocity } (\Delta \boldsymbol{v}) &= \text{final velocity} - \text{initial velocity}\\ \text{Final velocity} &= +8 \text{ m s}^{-1}\\ \text{Initial velocity} &= -10 \text{ m s}^{-1}\\ \text{Change in velocity} &= +8 \text{ m s}^{-1} - (-10 \text{ m s}^{-1}) = +18 \text{ m s}^{-1}\\ &\quad \text{(the positive means upward).}\end{aligned}$$

This is shown in Figure 3.11.

Figure 3.11

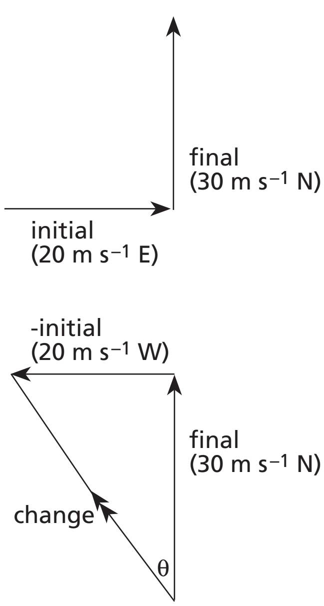


Example 2

A car travelling east at 20 m s^{-1} turns and accelerates to 30 m s^{-1} north (Figure 3.12). Calculate the change in velocity.

Solution

Figure 3.12



$$\begin{aligned}\text{Magnitude of change in velocity} &= \sqrt{20^2 + 30^2}\\ &= 36.1 \text{ m s}^{-1}\\ \text{Direction of change in velocity} &= \tan^{-1}\frac{20}{30} = 33.7°\end{aligned}$$

The change in velocity is 36.1 m s^{-1} in a direction N33.7°W

Questions

6 Calculate the change in velocity for each of these cases:

	Initial velocity	*Final velocity*
(a)	20 m s^{-1} south	30 m s^{-1} north
(b)	50 m s^{-1} west	10 m s^{-1} east
(c)	25 m s^{-1} north	35 m s^{-1} east
(d)	50 m s^{-1} south	20 m s^{-1} west

7 Determine the change in velocity of:

(a) a basketball with an initial velocity of 18 m s^{-1} down and a final velocity of 10 m s^{-1} up;

(b) a cricket ball travelling south at 30 m s^{-1} that struck a bat and was deflected to square leg (90°E of original path) at 25 m s^{-1};

(c) a bus travelling north at 20 km h^{-1}, which makes a 90° turn to the right without changing its speed.

8 A cricket ball delivered at 40 m s^{-1} strikes the pitch at 30° to the surface and bounces off the pitch at the same angle and speed. Calculate the change in velocity of the ball.

3.3 RESOLVING VECTORS INTO COMPONENTS

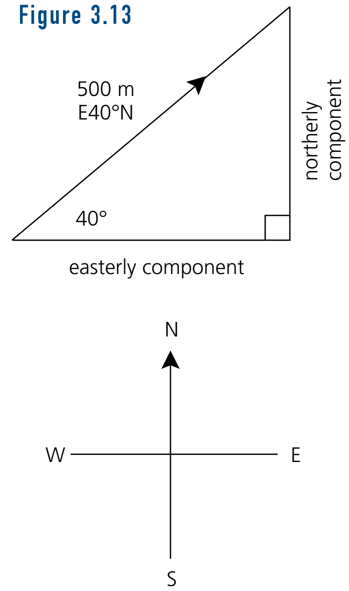
So far we have seen how two vectors can be added together to give a resultant third vector. The reverse process is called **resolution** (Latin *re* = 'back').

Why bother? In many cases it is convenient to 'break up' a vector into two **components** at right angles, for example vertically and horizontally. It is then sometimes easier to apply the laws of physics to the components.

Imagine that a person walked 500 m in a direction 40° to the north of east. This could be resolved into a northerly component and an easterly component:

$$\sin 40° = \frac{\text{northerly component}}{500\ \text{m}}$$
$$\text{Northerly component} = \sin 40° \times 500\ \text{m} = 321\ \text{m}$$
$$\text{Easterly component} = \cos 40° \times 500\ \text{m} = 383\ \text{m}$$

Figure 3.13



Example

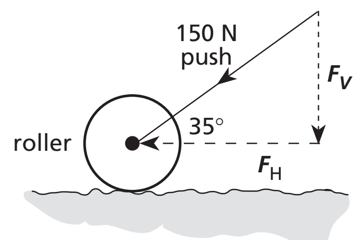
A roller is pushed along the ground with the handle at an angle of 35° to the horizontal. (See Figure 3.14.) Calculate **(a)** the horizontal component of the force pushing the roller over the ground; **(b)** the vertical component of the force pushing the roller into the ground. The push on the handle is 150 N.

Solution

$$\sin 35° = \frac{F_V}{150\ \text{N}} \text{ or } F_V = 150 \sin 35° = 150 \times 0.57 = 86\ \text{N}$$

$$\cos 35° = \frac{F_H}{150\ \text{N}} \text{ or } F_H = 150 \cos 35° = 150 \times 0.82 = 123\ \text{N}$$

Figure 3.14



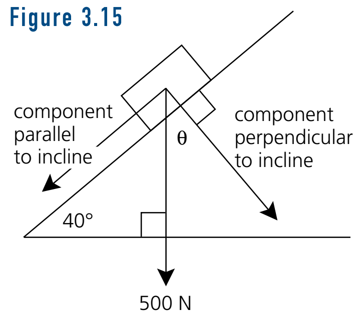
Questions

9 A girl pushes a shopping trolley along a horizontal path with a force of 100 N on the handle. If the angle between the handle and the ground is 30°, calculate the horizontal and vertical components of the pushing force.

10 A box of paper is being dragged along a vinyl floor by means of a rope angled at 20° to the floor. If an 80 N force is applied to the rope, calculate **(a)** the component of the force moving the box along the floor; **(b)** the component of the force tending to lift the box off the floor.

11 A 50 kg crate of rotting tomatoes rests on a 40° incline (Figure 3.15). If the force of gravity acting on the crate is 500 N vertically down toward the ground, calculate the components of the force **(a)** down the incline; **(b)** perpendicular to the incline. *Hint:* work out the value of θ first.

Figure 3.15



3.4 RELATIVE MOTION

When several motions are combined into one, some intriguing questions arise:

- In cricket, why does the bowler run to the wicket before delivering the ball?
- Why do you go slower when you row a boat upstream compared with rowing downstream?
- Why are jet planes launched off aircraft carriers into the wind?
- Why is it more dangerous to get out of a moving car than a stationary one?
- Why is a head-on collision worse than one with another car travelling in the same direction?

NOVEL CHALLENGE

A passenger on a train travelling at 60 km/h observes that it requires 4 s for another train 100 m long to pass her. *What is the speed of the second train?*

- When a person drives past you in a car you say they are moving. Couldn't they say you are moving and they are stationary? Who is right? Is this what physicists call **relative motion**?

The answers are very obvious, even without an understanding of vectors. But use of vectors can help us make predictions about the likely outcomes of such incidents.

Consider the first question about the cricketer. Imagine he could deliver a ball (B) from a standing position at 25 m s^{-1} relative to himself ($\mathbf{v}_{BC}$ = 25 m s^{-1}). If the cricketer runs at 5 m s^{-1} relative to the ground ($\mathbf{v}_{CG}$ = 5 m s^{-1}) while delivering it, then the ball would travel up the pitch at 30 m s^{-1}. This is a typical fast bowler's delivery speed. These motions can be expressed in equation form:

Velocity of ball relative to ground = velocity of ball relative to cricketer + velocity of cricketer relative to ground.

$$\mathbf{v}_{BG} = \mathbf{v}_{BC} + \mathbf{v}_{CG}$$

$$30 = 25 + 5$$

Note the order of symbols used in each case. When you write $\mathbf{v}_{BG}$, the first subscript (B) refers to a body in motion and the second letter (G) refers to whatever is being used for comparison. The second letter indicates the **frame of reference**, in this case the **G**round.

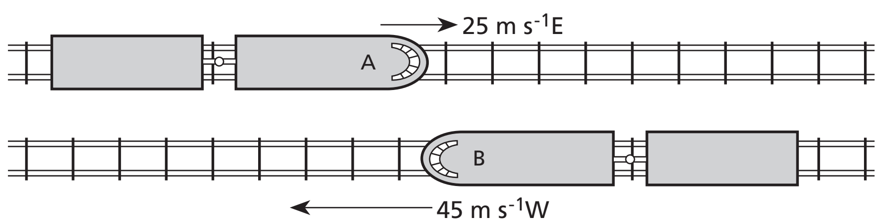
Note also that in the equation above the two inside subscripts on the right are the same letter (C). If this convention is always used then the two outside subscripts (B/G) are in the order that they should appear on the left.

$$\mathbf{v}_{BG} = \underbrace{\mathbf{v}_{BC} + \mathbf{v}_{CG}}_{\text{inside letter the same}}$$

Case 1: Parallel and toward each other

To illustrate relative motion further, consider two trains A and B moving toward each other on adjacent sets of tracks (Figure 3.16).

Figure 3.16



If train A is moving at 25 m s^{-1} to the right (relative to the ground) and train B is moving at 45 m s^{-1} to the left (relative to the ground) we can write their velocities as:

$\mathbf{v}_{AG}$ = +25 m s^{-1} $\quad$ $\mathbf{v}_{BG}$ = −45 m s^{-1} $\quad$ ('G' is the ground, the reference frame.)

The velocity of A relative to B is then:

$$\mathbf{v}_{AB} = \mathbf{v}_{AG} + \mathbf{v}_{GB} \quad \text{(using the conventions developed earlier)}$$

We don't have a value for $\mathbf{v}_{GB}$ but we do have a value for $\mathbf{v}_{BG}$. We can use the conversion:

$$\mathbf{v}_{GB} = -\mathbf{v}_{BG}$$

Hence

$$\begin{aligned}\mathbf{v}_{AB} &= \mathbf{v}_{AG} + \mathbf{v}_{GB}\\ &= \mathbf{v}_{AG} + -\mathbf{v}_{BG}\\ &= +25 \text{ m s}^{-1} + -(-45 \text{ m s}^{-1})\\ &= +70 \text{ m s}^{-1}\end{aligned}$$

You can think of this as being the same as train B being stationary and train A coming towards it at 70 m s^{-1}.

Note: you may prefer to assume all velocities are relative to the earth or ground (G). In this case the G term is often omitted and the relationship becomes $\boldsymbol{v}_{AB} = \boldsymbol{v}_A - \boldsymbol{v}_B$.

Case 2: Parallel and moving in the same direction

The example of the cricketer belongs to this type. Consider another example: Car X is moving along a highway at 100 km h^{-1} relative to the ground and passes car Y, which is travelling at 85 km h^{-1} relative to the ground. You should be able to find the speed of X relative to Y.

$\boldsymbol{v}_{XG} = +100 \text{ km h}^{-1}$ $\quad$ $\boldsymbol{v}_{YG} = +85 \text{ km h}^{-1}$

$\boldsymbol{v}_{XY} = \boldsymbol{v}_{XG} + \boldsymbol{v}_{GY} = \boldsymbol{v}_{XG} + -\boldsymbol{v}_{YG} = +100 \text{ km h}^{-1} + -(+85 \text{ km h}^{-1}) = +15 \text{ km h}^{-1}$

Questions

12 A girl is jogging at 4 m s^{-1} E and passes a boy who is jogging at 4 m s^{-1} W. What is the velocity of **(a)** the boy relative to the girl; **(b)** the girl relative to the boy; **(c)** the ground relative to the girl?

13 Why is it advantageous for planes to take-off into the wind?

14 A jet aircraft has an air speed of 720 km h^{-1} and is travelling the 4800 km west from Sydney to Perth. If the average wind speed is 40 km h^{-1} from the west, how long will it take the plane to reach Perth?

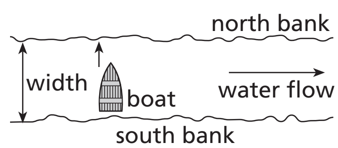
15 A boat that is capable of travelling at 4.5 km h^{-1} in still water is travelling on a river whose current is 1.5 km h^{-1} E. Find **(a)** the time it takes for the boat to travel 3.0 km downstream relative to the shore; **(b)** the time the boat takes to travel 3.0 km upstream against the current.

Case 3: Motion at right angles

The analysis of relative motion is fairly straightforward when the objects are moving parallel to each other. When other angles are involved, the situation becomes more complex.

Imagine a case where a person is rowing a boat *across* a river as shown in Figure 3.17. In this case the motion of the boat relative to the water is north and the motion of the water relative to the ground is west. The two motions are at right angles.

Figure 3.17



Example 1

A man can row a boat in still water at 30 m min^{-1}. He starts from the south shore of a river 600 m wide and aims due north. If the river is flowing west at 10 m min^{-1}, calculate:

(a) the velocity of the boat relative to the ground;
(b) the time taken to cross the river;
(c) the boat's landing position on the north shore.

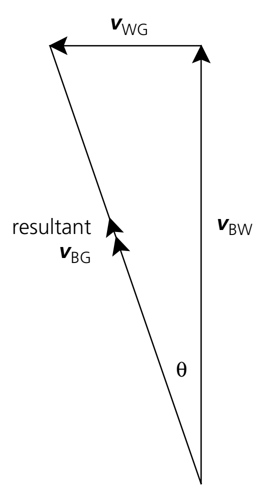
Solution

(a) $\boldsymbol{v}_{BG} = \boldsymbol{v}_{BW} + \boldsymbol{v}_{WG}$

$= 30 \text{ m min}^{-1} \text{ N} + 10 \text{ m min}^{-1} \text{ W}$

$\boldsymbol{v}_{BG} = \sqrt{(\boldsymbol{v}_{BW})^2 + (\boldsymbol{v}_{WG})^2} = 32 \text{ m min}^{-1}$; $\theta = 18°$ (N18°W).

Figure 3.18



(b) The boat is moving at 30 m min^{-1} towards the opposite bank, which is 600 m away. It doesn't matter that the current is dragging the boat sideways at the same time. The crossing time is independent of the river current.

$$t = \frac{s_1}{v_1} = \frac{600 \text{ m north}}{30 \text{ m min}^{-1} \text{ north}} = 20 \text{ min}$$

(c) The boat is carried downstream by the current. If the boat journey took 20 minutes and the current was flowing at 10 m min^{-1}, then the distance moved downstream is:

$$\boldsymbol{s}_2 = \boldsymbol{v}_2 t = 10 \text{ m min}^{-1} \text{ west} \times 20 \text{ min} = 200 \text{ m west}$$

Example 2

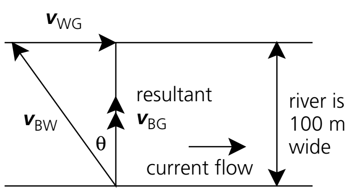
A person wishes to cross a 100 m wide river that is flowing east at 5 m s^{-1}. If they can row a boat in still water at 8 m s^{-1}, at what angle upstream should they head to end up on a point on the bank directly opposite? What is their crossing time?

Solution

In this case the resultant is not the hypotenuse of the vector triangle as is more common — it is the side adjacent to the angle θ.

Note that the convention about subscripts still holds for this diagram. The inside subscripts are the same (W):

Figure 3.19



$$\boldsymbol{v}_{BG} = \boldsymbol{v}_{BW} + \boldsymbol{v}_{WG}$$

$$\text{Using Pythagoras's theorem: } \boldsymbol{v}_{BG} = \sqrt{8^2 - 5^2} = 6.2 \text{ m s}^{-1}$$

$$\theta = \tan^{-1} 5/8 = 32°$$

$$\text{Crossing time: } t = \frac{s}{v} = \frac{100}{6.2} = 16 \text{ s}$$

Questions

16 A boat is driven with a velocity of 4.5 m s^{-1} relative to the water in a direction north straight across a river that is flowing at 2.0 m s^{-1} east.
(a) What is the boat's speed relative to the ground?
(b) If the river is 100 m wide, how far downstream will the boat reach the other side?

17 An airship is fitted with motors that can propel it at 6 km h^{-1} in still air. The captain wishes to travel 600 km due W but there is a wind blowing at 12 km h^{-1} from the east. Determine: **(a)** in what direction he must head; **(b)** what time his journey will take; **(c)** his ground speed.
If the wind was blowing at the same speed as before but from a direction of E30°S, determine: **(d)** in what direction he would now have to head; **(e)** what time his journey would now take; **(f)** his ground speed.

DRAWING GRAPHS 3.5

In the previous chapter you saw how graphing data was an important way of showing how motion varied as time passed. **Graphs** are a useful way of showing how one quantity depends on another.

On a graph the horizontal or *x*-axis is where the **independent variable** or cause is plotted. This also includes variables that progress regardless of the experiment. A good example is 'time elapsed', for time marches on whether any experiment is being carried out or not. The effect of that cause is plotted on the vertical or *y*-axis. This is called the **dependent variable**.

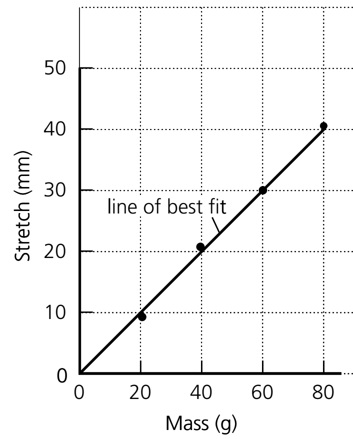
Linear relationships: direct proportion

Suppose, for example, that an experiment is performed to determine how much a certain rubber band stretches when masses are hung vertically from it. Data recorded from it are shown below:

mass (g)	0	20	40	60	80
stretch (mm)	0	9	21	30	42

The relationship becomes obvious when the points are plotted (Figure 3.20). Note that each point is plotted as a dot. You should put a circle around your dots. You should do this or use a cross rather than just a dot by itself because if the line joining the dots goes over the dots themselves they disappear. In this case a **line of best fit** is drawn. This has as many points on the line as possible. There are usually some points that aren't on the line and the line should be drawn so that there is an equal number below the line as above it. Scientists and engineers use a complex mathematical procedure (the method of least squares) to determine where the line of best fit should be. In some cases the line may not pass through any of the points. Any point that is a long way out of place is called an **outlier** and can be said to be **spurious**. It should be noted and the reasons for its existence be discussed but it should be left off the line.

Figure 3.20
Graph of linear relationship ($y \propto x$).



If the line is straight, as in the graph in Figure 3.20, it takes the general form of:

$$y \propto x$$

The proportional sign ($\propto$) can be replaced by an equals sign and a constant (m).

$$y = mx \quad \text{or} \quad y = mx + c$$

where x and y are the **variables**, m the **gradient** or **slope** of the line and c the **intercept** or point where the line cuts the y-axis.

In the graph of mass versus stretch, the intercept c is zero. The slope is found by dividing the change in y value by the change in the x value for the same section of the line. This can be written as:

$$\text{Slope } (m) = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Students often find it easier to remember this as 'rise over run' where *rise* refers to the y-axis and *run* refers to the x-axis.

The slope of the graph is given by: $m = \dfrac{40 - 0}{80 - 0} = 0.5$ and the intercept c is zero. This means that for every 1 g change in mass (x-axis), the rubber band changes by 0.5 mm in length (y-axis). Be careful with significant figures!

A straight line graph is said to be **directly proportional**. If it also passes through the **origin** (0,0) it is also said to be **linear**.

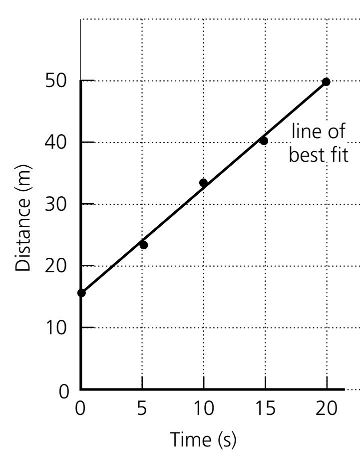
Example

The position of a car on a road is noted every 5 seconds and the following data obtained:

t = time elapsed (seconds)	0.0	5.0	10.0	15.0	20.0
s = displacement (metres)	16.0	23.5	34.0	41.0	50.0

(a) Plot the data.
(b) Calculate the average velocity (slope).
(c) State the intercept.
(d) State the equation for the line.
(e) Predict the position at 25.0 s.
(f) State the position at 15.0 s.

Figure 3.21
Graph of direct proportion ($y = mx + c$)



Solution

(a) See Figure 3.21.
(b) Slope = $\dfrac{50.0 - 16.0}{20.0 - 0.0} = 1.7 \text{ m s}^{-1}$.

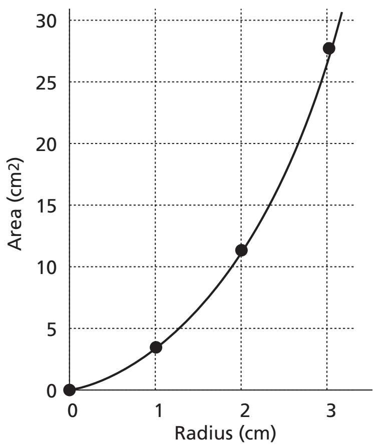
(c) Intercept = 16.0 m.
(d) $y = mx + c$, hence $s = 1.7t + 16$.
(e) The graph has to be extended to determine the position at 25.0 s. This is called **extrapolation**. Its value is approximately 58 m. Alternatively, the value $t = 25$ s could be substituted into the equation $s = 1.7t + 16$ to give the answer of 58.5 m.
(f) A value that is between two measured points is determined by **interpolation**. Its value is 37 m.

Non-linear relationships

The most common non-linear relationships you will meet in physics are:

- parabolic
- inverse
- inverse square
- exponential
- logarithmic.

Figure 3.22
Parabolic relationship ($y \propto x^2$).



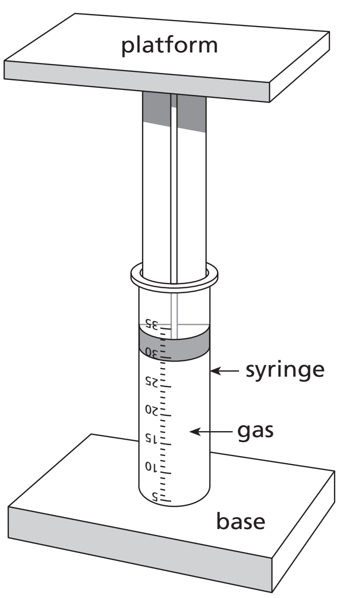
Parabolic relationships

Another direct relationship you will encounter is the parabolic relationship. The relationship between the area of a circle (A) and its radius (r) is a good example. The data below show these variables and Figure 3.22 shows a graph of the area as a function of the radius.

r = radius (cm)	0.0	1.0	2.0	3.0
A = area (cm^2)	0.0	3.1	12.6	28.3

Other phenomena which exhibit parabolic relationships are the paths of comets (except Halley's which is elliptical), curved mirrors in telescopes and projectiles (arrows in flight).

Figure 3.23
Boyle's law apparatus (see 'Inverse proportion').



Inverse proportion

Consider a case in which the volume of gas in a syringe is measured as the pressure on the syringe is increased (Figure 3.23):

P = pressure (kPa)	81	159	397	792
V = volume (mL)	10	5	2	1

When the data are plotted a curve like that shown in Figure 3.24 is obtained.
Note: 'data' is plural and should be followed by 'are'; a single data point is called a 'datum', which should be followed by 'is'. There is a tendency lately to use 'data' for both singular and multiple points. To be precise you should talk about 'these data' and not 'this data' but no one seems to care. It may not be important here but if your professional career involves writing technical reports, you may find your work is judged on simple grammar as much as anything else.

In the case above, the dependent variable (y-axis) decreases as the independent variable (x-axis) increases. In this case $y \propto \frac{1}{x}$ or $P \propto \frac{1}{V}$ and this is said to be **inversely proportional**. This relationship is commonly known as Boyle's law.

The proportional sign ($\propto$) can be replaced by an equal sign and a constant (k) so that the equation becomes $y = \frac{k}{x}$ or $P = \frac{k}{V}$. This implies that $P \times V$ is a constant. Examination of the above data will show that $P \times V$ is a constant and equals about 800. The values are slightly above and below 800 but these are within possible errors of an experiment.

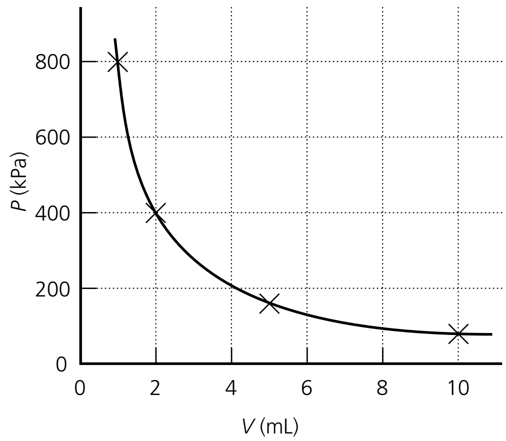


Figure 3.24
Graph of an inverse relationship ($y \propto 1/x$).

Other phenomena that exhibit parabolic relationships are the paths of comets (except Halley's, which is elliptical), curved mirrors in telescopes, and projectiles (arrows in flight).

Inverse square ($y = 1/x^2$)

This relationship looks similar to the inverse but has a much sharper bend. This type of relationship is very common in physics. For example, the variation in gravitational force with distance is given by $F \propto 1/d^2$. Other examples you will meet that vary in an inverse square relationship with distance are light intensity, centripetal force and electric field strength. Figure 3.25 shows how the magnetic force between two magnets varies with separation distance.

Exponential ($y = a^x$)

You will eventually meet some quantities in physics that are related exponentially. For example, the breakdown (decay) of a radioactive substance is given by: activity $\propto e^{-kt}$, where e and k are constants and t = time elapsed. (See Figure 3.26.)

Logarithmic ($y = \log_a x$)

A good example of this relationship is the response of the human ear (y) to sound energy (x). (See Figure 3.27.)

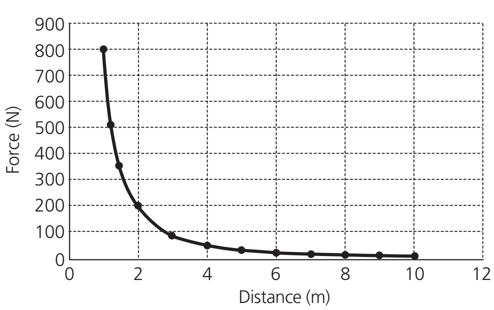


Figure 3.25
Graph of inverse square relationship ($y \propto 1/x^2$).

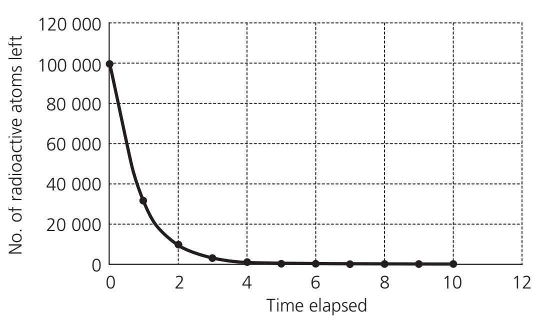


Figure 3.26
Graph of an exponential relationship ($y = a^x$).

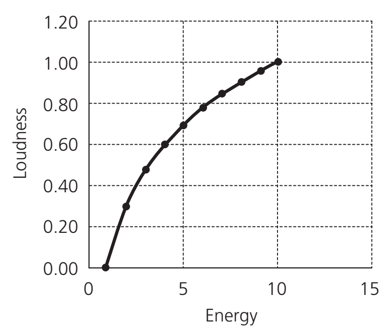


Figure 3.27
Graph of a logarithmic relationship ($y = \log_a x$).

Activity 3.2 MAKING A CUP OF TEA

SR

1 If you have an electric jug at home, determine the time it takes to boil 2 cups (500 mL) of tap water.

2 Look underneath the jug to see what power rating the jug is. It should be somewhere in the range of 1000 watts to 2000 watts.

3 Make a summary of the results of other people in your class and plot a graph of boiling time (y-axis) against power rating.

4 What is the relationship between the two measurements — are they inverse?

5 How could this experiment be improved to collect more reliable data?

Proving a relationship

You have seen that if $y \propto x$, then a graph of y versus x is a straight line. Similarly, if $y \propto \frac{1}{x}$, then a graph of y versus $\frac{1}{x}$ will also be a straight line.

Example 1

For the pressure and volume data on the previous pages, plot a graph of P vs $\frac{1}{V}$ to demonstrate that it is a straight line.

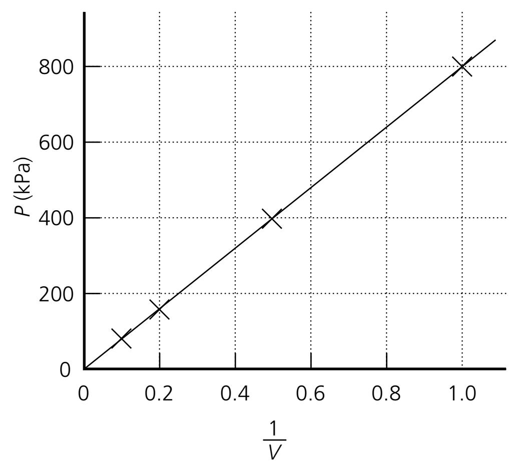
Solution

See Table 3.3 and Figure 3.28.

Table 3.3

P	81	159	397	792
V	10	5	2	1
$\frac{1}{V}$	0.1	0.2	0.5	1.0

Figure 3.28
Graph of P vs $\frac{1}{V}$ for a sample of gas.



Example 2

The displacement (s) in metres travelled by a car at various times t (in seconds) is shown below:

t	0	2	4	6	8
s	0	8	32	72	128

Draw a graph of **(a)** s vs t; **(b)** s vs t^2. What can you conclude?

Solution

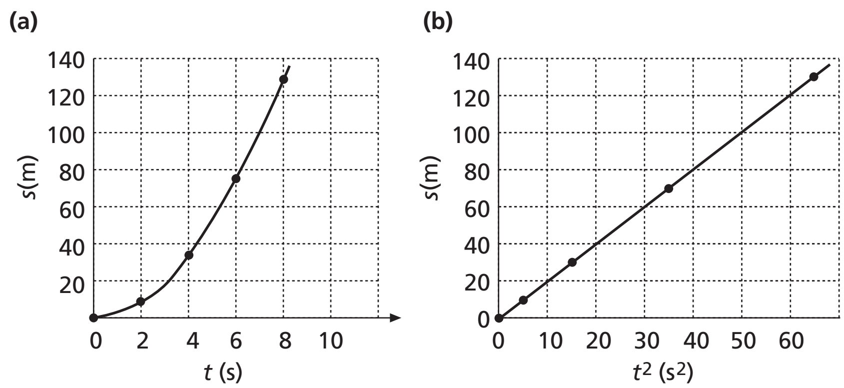


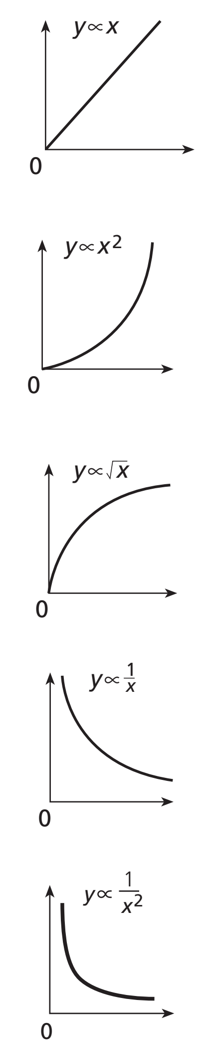
Figure 3.29

Conclude that s is proportional to t^2 as s vs t^2 is a straight line through the origin.

Working out relationships

The shapes of graphs often give a clue to the relationship between variables. Figure 3.30 shows five graphs and the relationship each suggests.

Figure 3.30 Five common relationships found in physics.



SR Activity 3.3 THE PLOT THICKENS

Part 1

If you have access to a computer with a spreadsheet you could construct a table that shows these relationships. You should fill the table with at least ten rows for each column and use a value of your choice for the gradient (m) — start with 2, and experiment.

For example, the formula for Cell D2 would be A2*A2 or A2^2.

Then use the plot, graph or chart function to see the shape of each relationship. You may need help to produce this spreadsheet.

	A	B	C	D	E	F
1	x-value	$y = mx$	$y = mx + c$	$y = x^2$	$y = 1/x$	$y = \sqrt{x}$
2	1					
3	2					
4	3					
5						

Figure 3.31 Computer spreadsheet.

You may even have access to a computer with graphing programs such as Sage or Omnigram.

Part 2

Try plotting sin, cos and tan of x for values of 0 to 360 degrees.

Part 3

What are the names of some computer applications that can be used to analyse data and come up with a mathematical relationship between variables? You may have to consult some computer magazines or professional journals such as *The Australian Physicist*, the *Journal of the Australian Institution of Engineers or Chemistry in Australia*.

Questions

18 If $W = kV$, then what is the effect on W of **(a)** tripling V; **(b)** halving V? What does a graph of W as a function of V look like?

19 For the graphs shown in Figure 3.32, select the graph that best represents:
(a) y is proportional to x; **(b)** y is inversely proportional to x;
(c) y is independent of x; **(d)** y is proportional to x^2.

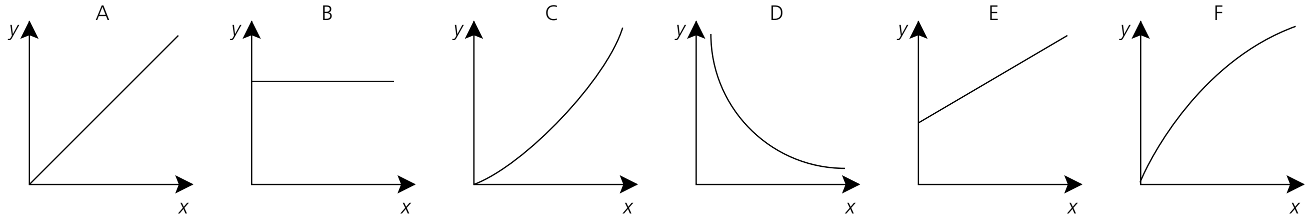


Figure 3.32
For question 19.

20 Plot a graph of each of the sets of data given below. In each case draw the line of best fit. (*Note:* the independent variable is listed first.)

(a)

Diameter of circle (cm)	0.0	4.0	8.0	12.0
Circumference of circle (cm)	0.0	12.5	25.4	37.3

(b)

Time (years)	0.0	1.0	2.0	3.0	4.0
Height of tree (m)	0.0	0.32	0.66	1.00	1.30

(c)

Time (s)	0.0	2.0	4.0	6.0
Distance (m)	0.0	12	23	37

21 For each of the lines plotted in the previous question, **(a)** calculate the slope; **(b)** extrapolate to 14 cm, 5.0 years and 8.0 seconds respectively; **(c)** interpolate for 6.0 cm, 2.5 y and 3.0 s respectively.

Practice questions

The relative difficulty of these questions is indicated by the number of stars beside each question number: * = low; ** = medium; *** = high.

Review — applying principles and problem solving

***22** Find the magnitude and direction of the resultant vector obtained by adding **(a)** displacements of 4.0 m E to 6.0 m W; **(b)** velocities of 5.0 m s^{-1} E and 5.0 m s^{-1} S; **(c)** accelerations of 3.2 m s^{-2} N and 4.8 m s^{-2} W; **(d)** displacements of 5.0 m W and 5.0 m NW.

***23** Calculate the following: **(a)** Add 15 m N, 23 m W and 20 m S. **(b)** Add 20 m E, 15 m N and 25 m N30°E. **(c)** Add 5 km NE, 20 km S and 15 m E. **(d)** Calculate 20 m N minus 18 m N.

****24** What is the change in velocity when:
(a) a tennis ball travelling at 80 km h^{-1} is hit directly back at a speed of 95 km h^{-1};
(b) a car travelling N at 45 km h^{-1} turns west and travels at 60 km h^{-1};
(c) a cricket ball strikes the pitch at a speed of 30 m s^{-1} and an angle to the ground of 28° and bounces up at an angle of 35° and a speed of 25 m s^{-1};
(d) a 4.5 g bullet travelling west at 700 m s^{-1} strikes a piece of armour plate and is deflected by 40° off course with a 20% loss in speed?

***25** What are the N and E components of **(a)** 100 km North; **(b)** 50 m s^{-1} N30°E; **(c)** 25 newton N40°E?

****26** A man can row a boat in a northerly direction at 5 m s^{-1} (relative to the water) across a river 300 m wide. A current is flowing due east at 12 m s^{-1}.
(a) What is the velocity of the boat relative to the ground?
(b) What time would it take to cross the river?
(c) How far downstream would the man land on the opposite bank?

27** For Figure 3.33, show by means of a sketch how you would **(a)** add the two vector quantities ***A and ***B***; **(b)** subtract vector ***A*** from vector ***B***; **(c)** multiply ***B*** by a factor of 3.

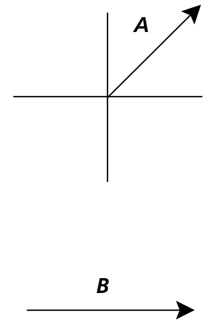


Figure 3.33
For question 27.

****28** Plot a graph to show the relationship between heat (***H***) in joule, developed in a heater in 10 minutes by electric currents of I ampere:

Current (I)	0.5	0.8	1.0	1.6	2.4	3.0
Heat (H)	375	960	1500	3840	8640	13 500

Plot a further graph to find the relationship between H and I.

****29** The table below shows the height and mass of the world's tallest man, Robert Wadlow, from birth to death at age 22 years. Plot these data on one graph and answer the questions that follow.

Table 3.4 ROBERT WADLOW'S GROWTH CHART

AGE (YEARS)	HEIGHT (cm)	MASS (kg)
0	45	3.85
5	163	48
8	183	77
9	189	82
10	196	95
11	200	—
12	210	—
13	218	116
14	226	137
15	234	161
16	240	170
17	245	143*
18	253	195
19	258	218
20	261	220
21	265	223
22.4**	272	199

* Following influenza.
** Died 15 July 1940 from a septic blister on his ankle.
(Source: *The Guinness Book of Records*)

(a) Calculate his fastest height growth rate. Include the units.
(b) What was his fastest mass growth rate?
(c) What was the cause of the negative slope in his mass growth rate? State this rate numerically.
(d) What were the average growth rates for height and mass over his lifetime?
(e) What would his height have been at age 16.5 years?
(f) If he had lived to age 23, predict his height.
(g) Why doesn't this graph pass through the origin?

Extension —complex, challenging and novel

*****30** A student whirls a red-brown rubber stopper of mass 50 g on the end of a nylon string in a horizontal clockwise circle of diameter 1.2 m (as seen from above) at a constant speed of 8 m s^{-1}. From an instant when the stopper is moving in a northerly direction, find its change in velocity after moving round **(a)** one-half of a revolution; **(b)** one-quarter of a revolution; **(c)** one-tenth of a revolution.

*****31** A 50 kg crate of winter clothing is pulled along a horizontal polished vinyl floor by means of a rope making an angle of 30° with the floor. If the pull in the rope is 100 N, calculate **(a)** the effective component of the force pulling the crate along the floor; **(b)** the component tending to lift the crate off the floor.

***32 A 55 year old pilot wishes to fly a 15 t Lockheed SR-71 jet plane to a place 250 km due east in 30 minutes. Find his air speed and course if there is a southerly wind blowing at 50 km h^{-1}.

***33 Two solid ball bearings P and Q are made of the same vanadium alloy with a density of 11.5 g cm^{-3}. The diameter of P is four times the diameter of Q. Write the mathematical relationship between **(a)** the surface area of P and the surface area of Q; **(b)** the volume of P and the volume of Q; **(c)** the mass of P and the mass of Q; **(d)** the density of P and the density of Q.

***34 The volume of one plastic sphere is 35 times the volume of a second sphere. **(a)** Write an equation showing the relationship between the radii of sphere 1 and the radii of sphere 2. **(b)** If the radius of the first sphere is 50 cm, find the radius of the second.

***35 A wooden ramp of mass 50 kg rises vertically 3.0 m for every 5.0 m of its length. A crate of salmon of weight 1000 N is placed on the ramp 2.0 m from the lower end. Find the component of the weight **(a)** parallel to the ramp; **(b)** perpendicular to the ramp.

***36 Water in a river 1.6 km wide flows at a speed of 6.0 km h^{-1}. A captain attempts to cross the river in his ferry at right angles to the bank but by the time it has reached the opposite bank the captain awakes and notices that it is 1.0 km downstream. If the captain wishes to take his boat directly across, what angle upstream must he point the boat assuming the boat speed remains the same?

***37 A coil of wire, which has a resistance of R and an inductance of L, has an impedance Z given by the relationship: $Z = \sqrt{R^2 + 4\pi^2 L^2 f^2}$ where f is the frequency of the AC electric current flowing through the coil of wire.

In an experiment to determine Z at a variety of frequencies, f, a researcher recorded the following data:

Frequency (Hz)	0	100	200	300
Impedance (ohms)	100	121	162	210

By drawing the appropriate graph or other means, determine the values of R and L for this component.

***38 German physicist Arnd Leike, from the University of Munich, found that the decay of foam height in beer with time was exponential: $y \propto 1/x^n$, where y = height of the foam and x = time. He was awarded an 'Ig Nobel' Prize by the science humour magazine *Annals of Improbable Research* for one of the world's most useless pieces of research. Using an Excel spreadsheet or your graphing calculator, describe the difference between the graphs when $n = 3$ (Leike's result) and $n = 2$ (inverse square).